

HYPERPLANES IN ABELIAN GROUPS AND TWISTED SIGNATURES

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ABSTRACT. We investigate the following question: if A and A' are products of finite cyclic groups, when does there exist an isomorphism $f : A \rightarrow A'$ which preserves the union of coordinate hyperplanes (equivalently, so that $f(x)$ has some coordinate zero if and only if x has some coordinate zero)?

We show that if such an isomorphism exists, then A and A' have the same cyclic factors; if all cyclic factors have order larger than 2, the map f is diagonal up to permutation, hence sends coordinate hyperplanes to coordinate hyperplanes.

As a model application, we show using twisted signatures that there exists a family of compact 4-manifolds $X(n)$ with $H_1 X(n) = \mathbb{Z}/n$ with the property that $\prod X(n_i) \cong \prod X(n'_j)$ if and only if the factors may be identified (up to permutation), and that the induced map on first homology is represented by a diagonal matrix.

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1. INTRODUCTION

Let $A = \prod_{i=1}^m \mathbb{Z}/n_i$ be a product of non-trivial finite cyclic groups. We say that

$$Z_i = \{(x_1, \dots, x_m) \in A \mid x_i = 0\}$$

is the i 'th coordinate hyperplane (the set of points where the i 'th coordinate is zero). Their union $\bigcup_{i=1}^m Z_i$ is the set of elements $x \in A$ for which at least one coordinate x_i is zero, which we denote $\mathcal{Z}(A)$.

Taking Z_i as a model example of a hyperplane, we define a hyperplane in an abelian group to be a subgroup with non-trivial cyclic quotient. Our first interest is the following informal question: *When can one recover a set of hyperplanes from their union?*

When A is a vector space, the corresponding question is relatively straightforward. It is visually clear that the only hyperplanes contained in $\mathcal{Z}(\mathbb{R}^m)$ are the coordinate hyperplanes. Some linear algebra shows that the same is true for $\mathcal{Z}((\mathbb{Z}/p)^m)$ for $p > 2$.

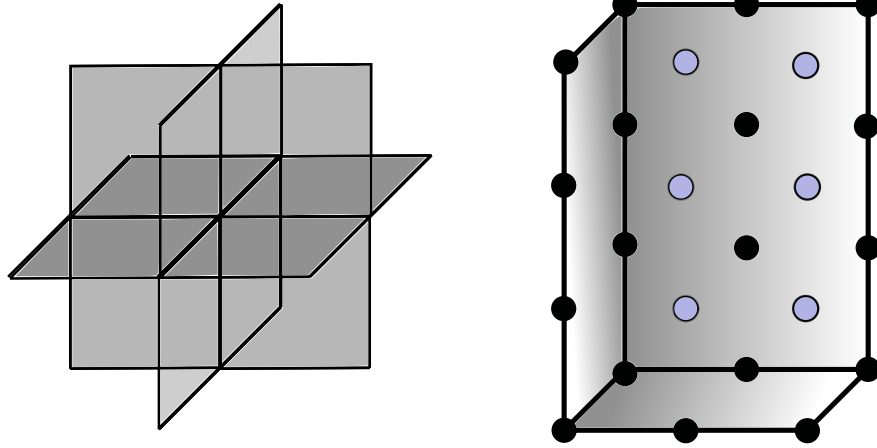


FIGURE 1. $\mathcal{Z}(\mathbb{R}^3)$ and $\mathcal{Z}(\mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/4)$. In the latter, we color in the points on $\mathcal{Z}$ but leave gray the points in its complement.

However, the corresponding claim is false for $A = (\mathbb{Z}/2)^m$ as soon as $m \geq 3$ (or $m \geq 2$ if we allow affine hyperplanes): we have a containment of the affine line

$$\{(x, y) \mid x + y = 1\} = \{(1, 0), (0, 1)\} \subset \mathcal{Z}((\mathbb{Z}/2)^2),$$

and this hyperplane is maximal with respect to inclusion among hyperplanes contained in $\mathcal{Z}(A)$, but is certainly not one of the coordinate hyperplanes.

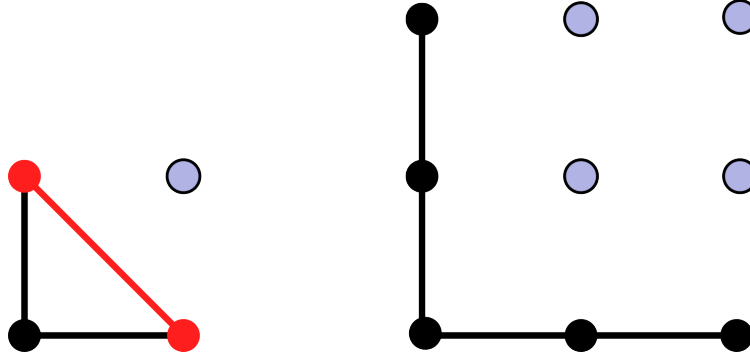


FIGURE 2. The only affine lines contained in $\mathcal{Z}((\mathbb{Z}/3)^2)$ are the coordinate lines. By contrast, $\mathcal{Z}((\mathbb{Z}/2)^2)$ contains the affine line $\{(1, 0), (0, 1)\}$.

If we restrict ourselves to linear hyperplanes, the first counterexample is the hyperplane

$$\{(x, y, z) : x + y + z = 0\} = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\} \subset \mathcal{Z}((\mathbb{Z}/2)^3).$$

The question is further complicated when A is not a vector space, as our definition allows for nested hyperplanes; for instance, $2\mathbb{Z}/30 \subset 6\mathbb{Z}/30$ are both non-trivial hyperplanes in $\mathbb{Z}/30$, as both have cyclic quotient.

To attempt to recover the set of coordinate hyperplanes from their union, we should determine what's special about this set. We propose in Definition 3 the notion of a *hyperplane splitting*, a set $\{H_1, \dots, H_m\}$ of hyperplanes so that the natural map $\pi : A \rightarrow \prod_{i=1}^m A/H_i$ is

an isomorphism. Our first result is that — provided one assumes there are no $\mathbb{Z}/2$ factors — one can indeed recover a hyperplane splitting from its union.

Theorem 1. *Let $A = \prod_{i=1}^m \mathbb{Z}/n_i$ be a product of finite cyclic groups, with $n_i > 2$ for all i . If $\{H_1, \dots, H_{m'}\}$ is any hyperplane splitting of A with union $\bigcup_{i=1}^{m'} H_i = \mathcal{Z}(A)$, then $m = m'$ and the H_i are the coordinate hyperplanes of A .*

A more general version of this statement allowing for $n_i = 2$ is proved as Theorem 12 below. One can recover the number of coordinate hyperplanes, and those of odd order quotient are necessarily coordinate hyperplanes. On the other hand, the even order H_i may be more complicated ‘nearly-coordinate hyperplanes’ (which can only appear when at least one factor is $\mathbb{Z}/2$) modeled on the examples of interesting non-coordinate hyperplanes in $\mathcal{Z}((\mathbb{Z}/2)^m)$ above.

As a corollary of our more general statement on hyperplane splittings with given union, we have the following theorem on isomorphisms which preserve the union of coordinate hyperplanes.

Theorem 2. *Suppose*

$$A = (\mathbb{Z}/2)^\ell \times \prod_{i=\ell+1}^m \mathbb{Z}/n_i \text{ and } A' = (\mathbb{Z}/2)^{\ell'} \times \prod_{j=\ell'+1}^{m'} \mathbb{Z}/n'_j,$$

where $n_i, n'_j > 2$ for $i > \ell$ and $j > \ell'$. If $f : A \rightarrow A'$ is an isomorphism with $f(\mathcal{Z}(A)) = \mathcal{Z}(A')$, then we have $\ell = \ell'$, $m = m'$; additionally, up to a permutation of the n'_j , we have $n_i = n'_i$ and a matrix representative of f takes the form

$$f = \begin{pmatrix} f_{11} & 0 \\ f_{21} & D \end{pmatrix},$$

where f_{11} is an isomorphism and D is a diagonal matrix.

We prove this by examining a special family of hyperplanes in $\mathcal{Z}(A)$, those containing an element with all but one coordinate equal to 1. Such hyperplanes are particularly tractable, and classifying them reduces to the case of p -groups for p a prime and one factor cyclic of order p . When p is odd, such hyperplanes are coordinate hyperplanes; when $p = 2$, the aforementioned nearly-coordinate hyperplanes make an unavoidable appearance.

Our initial interest was driven in part by cancellation phenomena for certain types of manifold invariants. Suppose one has a mathematical object X together with a natural assignment of a group $H(X)$ and a function $\phi_X : H(X) \rightarrow \mathbb{C}$. Suppose that one has a product operation $X \boxtimes Y$ on these mathematical objects (usually Cartesian product or connected sum) so that $H(X \boxtimes Y) = H(X) \times H(Y)$. We say that ϕ is *additive* if $\phi_{X \boxtimes Y}(h_1, h_2) = \phi_X(h_1) + \phi_Y(h_2)$, while we say that ϕ is *multiplicative* if $\phi_{X \boxtimes Y}(h_1, h_2) = \phi_X(h_1) \cdot \phi_Y(h_2)$.

The main theorem of [Mil22] shows that when ϕ is **additive**, provided the ϕ_{X_i} distinguish the X_i and each has a certain non-vanishing property, one may recover the ϕ_{X_i} from the invariant $\phi_{\boxtimes X_i}$. This was applied to classify connected sums of lens spaces up to integer homology cobordism by taking X_i to be 3-dimensional lens spaces, $\boxtimes$ to be connected sum, and ϕ to be the Heegaard Floer d -invariant.

The argument in the additive case reduces to showing that one can recover the coordinate lines in an abelian group from their union. To make an analogous argument in the multiplicative setting, one is led to precisely Theorem 2. As a model application, we investigate one particular multiplicative invariant: twisted signatures of 4-manifolds.

Associated to a compact oriented 4-manifold X and a homomorphism $\phi : H_1(X) \rightarrow S^1$ is its *twisted signature*, an integer $\sigma(X, \phi)$; we review these in Section 5. The corresponding untwisted signature $\sigma(X, 1)$ is abbreviated to $\sigma(X)$.

We say that a compact connected oriented 4-manifold is *signature-simple* if $\sigma(X, \phi) = 0$ if and only if ϕ is the trivial homomorphism. We say a *signature-simple family* is a choice, for each **odd** integer $n > 1$, of a signature-simple manifold $X(n)$ with $H_1(X) \cong \mathbb{Z}/n$. A (non-constructive) proof that such manifolds exist for each odd n is given in Section 5 below.

Theorem 3. *Suppose $X(n)$ is a signature-simple family. If there is a homeomorphism*

$$f : \prod_{i=1}^m X(n_i) \rightarrow \prod_{j=1}^{m'} X(n'_j) \times Y,$$

where Y is a compact connected manifold with $\sigma(Y) \neq 0$ and $H_1(Y) = 0$, then $m = m'$ and $Y = \{*\}$; further, up to a permutation of the n'_j , we have $n_i = n'_i$ and f induces a diagonal map on first homology.

The set of $\phi \in \text{Hom}(H_1 X, S^1) \cong \prod_{i=1}^m \mathbb{Z}/m_i$ for which the twisted signatures of the products vanish can be identified with the set of homomorphisms which are trivial on at least one factor. Because a homeomorphism f must preserve these twisted signatures, it follows that it satisfies the assumptions of Theorem 2, with all $n_i > 2$.

Organization. In Section 2, we introduce hyperplane splittings and nearly-coordinate hyperplanes, as well as their affine cousins, which appear naturally in our inductive argument. In Section 3, we classify a particular family of affine hyperplanes contained in $\mathcal{Z}(A)$ (“ $\tilde{z}_0$ -hyperplanes”), first reducing to the case of p -groups — which are especially amenable to inductive argument — and then handling the general case. In Section 4, we use our classification of $\tilde{z}_0$ -hyperplanes to prove Theorems 1-2. Finally, in Section 5 we explain how to apply these results to twisted signatures, and give a construction of signature-simple manifolds.

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2. AFFINE SUBGROUPS AND HYPERPLANES

In what follows, we will frequently want to refer to terms like $\phi(1, \dots, 1)$, and sometimes the ambiguous length of the string can cause confusion. We write $\vec{1}^m$ for a string of m ones. As is standard, we write $\vec{e}_i$ for the i ’th standard basis vector; its length will be clear from context.

In the introduction, we discussed hyperplanes contained in $\mathcal{Z}(A)$, where $A = \prod_{i=0}^m \mathbb{Z}/n_i$ is a product of finite abelian groups.¹ If $A' = \prod_{i=0}^{m-1} \mathbb{Z}/n_i$, our inductive approach has us pass from a hyperplane in A to its intersection with $A' \times \{1\}$ projected to A' ; the new subset H_1 then satisfies $H_1 \subset \mathcal{Z}(A')$. However, because $A' \times \{1\}$ is not a subgroup, neither is H_1 . Rather, we must investigate the more general notion of *affine hyperplanes*. To start, we should define an affine subgroup.

Definition 1. *Let A be an abelian group. An **affine subgroup** in A is a **nonempty** subset $S \subset A$ satisfying any of the following three equivalent conditions.*

- (a) $S = x + S'$ is a coset of a subgroup $S' \subset A$. We call S' the **linear translate** of S .
- (b) $S = f^{-1}(c)$ for some group homomorphism $f : A \rightarrow B$.
- (c) S is closed under affine combinations $\sum_{i=1}^k t_i x_i$, where $x_i \in S$ and t_i are integers with $\sum_{i=1}^k t_i = 1$.

If $S = S'$ (equivalently, if $0 \in S$), we say that S is a **linear subgroup**.

It is straightforward and somewhat pleasant to verify that these conditions are equivalent, and we leave this to the reader. While a ‘linear subgroup’ is by definition a subgroup, the superfluous adjective is useful to contrast with the affine case.

Given an example of an affine subgroup, one can quickly produce more: the image of an affine subgroup under a homomorphism is affine, as is the preimage of an affine subgroup (provided it is nonempty). Similarly, a nonempty intersection of two affine subgroups is again affine.

Definition 2. *Let A be an abelian group.*

- A **linear hyperplane** in A is a proper subgroup $H \subset A$ with cyclic quotient; equivalently, the zero set of a non-trivial homomorphism $f : A \rightarrow \mathbb{R}/\mathbb{Z}$.
- An **affine hyperplane** in A is a coset of a linear hyperplane; equivalently, a nonempty set of the form $f^{-1}(c)$ where $f : A \rightarrow \mathbb{R}/\mathbb{Z}$ is a non-trivial homomorphism.

We allow for a slightly more general notion of hyperplane splitting than discussed in the introduction, to allow for affine hyperplanes. While this is not relevant in our final application, our arguments apply to the affine context without change.

Definition 3. *A **hyperplane splitting** of A is an unordered collection $\mathcal{H} = \{H_0, \dots, H_m\}$ of affine hyperplanes of A so that the induced map $f : A \rightarrow \prod_{i=0}^m A/H_i'$ is an isomorphism. We write $U(\mathcal{H}) = \bigcup_{i=0}^m H_i$ for the **hyperplane union**.*

The standard example of a hyperplane splitting is the one introduced at the beginning of this article: $A = \prod_{i=0}^m \mathbb{Z}/n_i$ has a hyperplane splitting $\{Z_0, \dots, Z_m\}$ with Z_j the j 'th coordinate hyperplane, the set of points whose j 'th coordinate is zero. In fact, given an arbitrary hyperplane splitting of an arbitrary finite abelian group B , write n_i for the order of B/H_i' ; then the homomorphism

$$\pi : B \rightarrow \prod_{i=0}^m B/H_i' \cong \prod_{i=0}^m \mathbb{Z}/n_i$$

sends each H_i isomorphically onto a translate of the corresponding coordinate hyperplane. An appropriate translation gives rise to an affine isomorphism $\pi + c : B \cong \prod_{i=0}^m \mathbb{Z}/n_i$ which

¹Note that we now index our products starting at $i = 0$, unlike in the introduction; this choice cleans up the notation in the crucial Definition 4 of nearly-coordinate hyperplanes.

sends each H_i to Z_i . Thus every hyperplane splitting is isomorphic to the standard splitting by coordinate hyperplanes.

In a hyperplane splitting $\mathcal{H}$, the H_i are *maximal* affine hyperplanes contained in $U(\mathcal{H})$:

Lemma 4. *If $\mathcal{H}$ is a hyperplane splitting of A , the H_i are affine hyperplanes contained in $U(\mathcal{H})$, and they are maximal with respect to this property.*

Proof. It suffices to verify this for $\mathcal{H}$ the coordinate hyperplane splitting, where it is clear. $\square$

It would be reasonable to guess that the coordinate hyperplanes are precisely the maximal hyperplanes contained in $\mathcal{Z}(A)$, or even better, that they are the maximal *affine* hyperplanes contained in $\mathcal{Z}(A)$ — so that a maximal affine hyperplane is in fact a linear hyperplane. As discussed in the introduction, some simple computation shows that this is true over $(\mathbb{Z}/p)^m$ for $p > 2$, but false for $p = 2$, with model example being those vile hyperplanes

$$V_m = \{(x_0, \dots, x_m) \mid x_0 + \dots + x_m = m\} \subset (\mathbb{Z}/2)^{m+1}$$

for $m > 0$. The hyperplane V_m lies in $\mathcal{Z}((\mathbb{Z}/2)^{m+1})$, which is the complement of the single element $\vec{1}^{m+1} \in (\mathbb{Z}/2)^{m+1}$; further, V_m is maximal with respect to this property.

There are three ways to construct more hyperplanes with properties similar to the V_m .

- One can add a new, independent, factor to form

$$V_m \times \mathbb{Z}/n \subset (\mathbb{Z}/2)^{m+1} \times \mathbb{Z}/n.$$

- One can *inflate a coordinate* by a factor of r to form the affine hyperplane

$$\{(rx_0, \dots, x_m) \mid x_0 + \dots + x_m = m\} \subset (\mathbb{Z}/2r) \times (\mathbb{Z}/2)^m$$

with quotient isomorphic to $\mathbb{Z}/2r$. While this construction can be iterated (performed on different coordinates) and returns another affine hyperplane so long as the inflation factors are pairwise coprime, we will soon restrict attention to a particular family of hyperplanes chosen to exclude any such doubly-inflated examples.

- After applying the first construction any number of times and the second construction once, one can permute the coordinates of A .

We can rewrite the hyperplane $V_m \times \prod_{i=m+1}^k \mathbb{Z}/n_i$ with k independent factors as the set of elements satisfying

$$x_0 = \phi(x_1, \dots, x_{m+k}) + \phi(\vec{1}^{m+k}),$$

where $\phi : (\mathbb{Z}/2)^m \times \prod_{i=1}^k \mathbb{Z}/n_i \rightarrow \mathbb{Z}/2$ is given by $\phi(x_1, \dots, x_{m+k}) = x_1 + \dots + x_m$. Taking into account inflation and permutation, this leads us to the following definition.

Definition 4. *Consider the product of finite cyclic groups $\mathbb{Z}/2r \times \prod_{i=1}^m \mathbb{Z}/n_i$, with $r \geq 1$ and $n_i \geq 2$. A **nearly- Z_0 hyperplane** is an affine hyperplane obtained by the following construction. Choose a homomorphism $\phi : \prod_{i=1}^m \mathbb{Z}/n_i \rightarrow \mathbb{Z}/2$ which is nonzero on $\vec{e}_i$ only if $n_i = 2$ (we say ϕ is supported on $\mathbb{Z}/2$ factors). Then the corresponding nearly- Z_0 hyperplane is*

$$Z_\phi = \{(x_0, \dots, x_m) \mid x_0 = r(\phi(x_1, \dots, x_m) + \phi(\vec{1}^m))\}$$

We say that x_0 is the **determined coordinate**, and if $\phi(\vec{e}_i) \neq 0$ we say x_i is a **determining coordinate**. If $\phi(\vec{e}_i) = 0$, we say x_i is an **independent coordinate**.

A hyperplane obtained from Z_ϕ by permuting the coordinates such that x_0 is sent to x_j is called a **nearly- Z_j hyperplane**. If a hyperplane is nearly- Z_j for some j , we say it is **nearly-coordinate**.

Remark 1. When $\phi = 0$, we have (as the notation suggests) $Z_\phi = Z_0$. In particular, when all $n_i > 2$, the nearly-coordinate hyperplanes are simply the coordinate hyperplanes.

Remark 2. When $r > 1$, the determined coordinate of a nearly-coordinate hyperplane is well-defined: the determined coordinate is the unique coordinate x_i so that the i 'th projection has $\pi_i(H) \neq \mathbb{Z}/n_i$. However, when $r = 1$, it is impossible to distinguish the role of the determined coordinate and the determining coordinates; permuting the determined coordinate with some determining coordinate returns the same nearly-coordinate hyperplane.

The nearly-coordinate hyperplanes can be thought of as hyperplanes where, instead of demanding that $x_i = 0$, we demand that x_i is determined by the values in the other $\mathbb{Z}/2$ -coordinates. The addition of a $\phi(\vec{1}^m)$ term is to ensure that $\vec{1}^{m+1}$ does not lie in Z_ϕ , which ultimately guarantees that $Z_\phi \subset \mathcal{Z}(A)$. In fact, we have better:

Lemma 5. *Any nearly-coordinate hyperplane in A is a hyperplane, is contained in $\mathcal{Z}(A)$, and is maximal with respect to these properties.*

Proof. The stated properties are invariant under permutations, so it suffices to check them for nearly- Z_0 hyperplanes Z_ϕ . By definition, Z_ϕ is the nonempty inverse image of $r\phi(\vec{1}^m)$ under the homomorphism $f_\phi : A \rightarrow \mathbb{Z}/2r$ defined by

$$f_\phi(x_0, \dots, x_m) = x_0 - r\phi(x_1, \dots, x_m),$$

so Z_ϕ is an affine hyperplane. To see that $Z_\phi \subset \mathcal{Z}(A)$, observe that if all coordinates x_i for $i > 0$ are non-trivial, then $\phi(x_1, \dots, x_m) = \phi(\vec{1}^m)$; the only coordinates which contribute to the value of ϕ are the $\mathbb{Z}/2$ coordinates, where $x_i \neq 0$ means precisely that $x_i = 1$. Thus if

$$x_0 = r \left(\phi(x_1, \dots, x_m) + \phi(\vec{1}^m) \right)$$

where all $x_i \neq 0$ for $i > 0$, we must have $x_0 = 0$; we have established every element of Z_ϕ has some coordinate equal to zero, as desired.

As for maximality, suppose $Z_\phi \subsetneq H$ is properly contained in a larger hyperplane; we will show H contains some element with all entries nonzero. Pick $(y_0, \vec{y}) \in H \setminus Z_\phi$, meaning that $y_0 \neq r(\phi(\vec{y} + \vec{1}^m))$. Observe that there is a distinct vector $(x'_0, \vec{y}) \in Z_\phi \subset H$ with $x'_0 = r(\phi(\vec{y} + \vec{1}^m))$, and that $(0, \vec{1}^m) \in Z_\phi \subset H$ by the definition of Z_ϕ . Taking affine combinations of these vectors, we see that

$$(y_0 - x'_0, \vec{1}^m) = (y_0, \vec{y}) - (x'_0, \vec{y}) + (0, \vec{1}^m) \in H,$$

which has no component zero, as $y_0 \neq x'_0$. □

Further, at least in a simple case, we can verify that nearly-coordinate hyperplanes are the only affine hyperplanes in $\mathcal{Z}(A)$.

Lemma 6. *Every affine hyperplane contained in $\mathcal{Z}(A)$ with quotient of order 2 is a nearly-coordinate hyperplane.*

Proof. Let $f : A \rightarrow \mathbb{Z}/2$ be a non-trivial homomorphism with $H = f^{-1}(c)$ contained in $\mathcal{Z}(A)$. Because the claimed properties are invariant under permutation, we suppose that f is supported in precisely the coordinates $0 \leq i \leq j$, which must have n_i even and $f(\vec{e}_i) = 1$. In particular, $f(\vec{1}^{m+1}) = j + 1$. Because $H \subset \mathcal{Z}(A)$, the element $\vec{1}^{m+1} \notin H$, so $f(\vec{1}^{m+1}) \neq c$; we must have $c = j$.

Now for each $0 \leq i \leq j$, we have $f(\vec{1}^{m+1} + \vec{e}_i) = j + 2 = c$, implying $\vec{1}^{m+1} + \vec{e}_i \in \mathcal{Z}(A)$. Because this vector has i 'th coordinate 2 and all other coordinates 1, the only coordinate which can be zero is the i 'th; we must have $n_i = 2$ for all $0 \leq i \leq j$. Thus

$$\begin{aligned} H = f^{-1}(c) &= \{(x_0, \dots, x_m) \in A \mid x_0 + \dots + x_j = j\} \\ &= \{(x_0, \dots, x_m) \in A \mid x_0 = x_1 + \dots + x_j + j\}. \end{aligned}$$

This is precisely Z_ϕ for $\phi(x_1, \dots, x_m) = x_1 + \dots + x_j$. $\square$

Before moving on, let us point out that the *hyperplane* hypothesis is significant.

Example 3. The group $(\mathbb{Z}/n)^n$ contains the affine subgroup whose elements are cyclic permutations of $(0, 1, \dots, n-1)$, in which all terms have some nonzero coordinate, but for $n > 2$ this is not contained in anything comparable to a nearly-coordinate hyperplane. Notice that the quotient of the corresponding linear hyperplane is isomorphic to $(\mathbb{Z}/n)^{n-1}$, which is cyclic only for $n = 2$, where this construction gives the nearly-coordinate hyperplane V_1 .

The hyperplane condition allows us to rule out these non-examples. Notice that in all of these affine subgroups, **exactly** one coordinate of each element is zero. We will investigate hyperplanes containing a specific vector with that property: the vector

$$\vec{z}_0 = \vec{1}^{m+1} - \vec{e}_0 = (0, \vec{1}^m),$$

which appeared in the proof of Lemma 5. This will allow us to set up an induction, but it also reduces the number of potential counter-examples in the previous family:

Example 4. When $n = p$ is prime, by rescaling all coordinates after the first we may obtain an affine subgroup of $(\mathbb{Z}/p)^p$ which contains $\vec{z}_0$ and is contained in $\mathcal{Z}((\mathbb{Z}/p)^p)$. For instance, for $(\mathbb{Z}/3)^3$, scaling by $(1, 1, 1/2) = (1, 1, 2)$ one obtains the affine subgroup $\{(0, 1, 1), (2, 0, 2), (1, 2, 0)\}$. If we rescale this affine subgroup of $(\mathbb{Z}/5)^5$ pointwise by the scalars $(1, 1, 1/2, 1/3, 1/4) = (1, 1, 3, 2, 4)$, we obtain the affine subgroup

$$\{(0, 1, 1, 1, 1), (4, 0, 3, 4, 2), (3, 4, 0, 1, 3), (2, 3, 2, 0, 4), (1, 2, 4, 3, 0)\}.$$

3. $\vec{z}_0$ -HYPERPLANES

As we will see in Section 4, our main results will follow from the classification of affine hyperplanes containing the element $\vec{z}_i = \vec{1}^{m+1} - \vec{e}_i$. Throughout the section, we write $A = \prod_{i=0}^m \mathbb{Z}/n_i$ unless stated otherwise.

Definition 5. A $\vec{z}_i$ -**subgroup** in A is an affine subgroup S which has $\vec{z}_i \in S \subset \mathcal{Z}(A)$. A $\vec{z}_i$ -**hyperplane** is a $\vec{z}_i$ -subgroup with A/S' cyclic.

The reader should keep in mind that each $\vec{z}_i$ -subgroup is contained in $\mathcal{Z}(A)$, even though this property is not mentioned in the name.

If a hyperplane splitting has $U(\mathcal{H}) = \mathcal{Z}(A)$, then for each i some hyperplane in the splitting must contain $\vec{z}_i$. Further, such a vector $\vec{z}_i$ is unique so long as we avoid order 2 hyperplanes.

Lemma 7. Given a $\vec{z}_i$ -subgroup S with $n_i > 2$, no other $\vec{z}_j$ is contained in S .

Proof. Permuting coordinates, it suffices to show that if S is both a $\vec{z}_0$ - and $\vec{z}_1$ -subgroup, then $n_0 = 2$. Supposing $\vec{z}_0, \vec{z}_1 \in S$ and taking affine combinations, we see that

$$(2, -1, \vec{1}^{m-1}) = 2\vec{z}_1 - \vec{z}_0 \in S.$$

Because $S \subset \mathcal{Z}(A)$, we must have $2 = 0$ in $\mathbb{Z}/n_0$, and hence $n_0 = 2$. $\square$

As in the above argument, we may freely permute coordinates and focus our attention on $\vec{z}_0$ -hyperplanes. Our main result in this section, and the main technical result in this paper, is that the **maximal** $\vec{z}_0$ -hyperplanes are heavily constrained:

Theorem 8. *Every $\vec{z}_0$ -hyperplane is contained in a nearly- Z_0 hyperplane. In particular, the **maximal** $\vec{z}_0$ -hyperplanes are precisely the nearly- Z_0 hyperplanes.*

We prove this theorem in three steps, each slightly more general than the last.

First, in Section 3.1, we prove the corresponding claim when $n_0 = p$ is prime and $n_i = p^{j_i}$ for $i > 0$. This is especially tractable, as if one finds that the quotient A/H' contains two $\mathbb{Z}/n_i$ factors, it is automatically non-cyclic (as these factors are never coprime). In turn, the corresponding version of Theorem 8 gives an **equality** instead of a containment.

Next, in Section 3.2, we handle the case that the first factor is still $\mathbb{Z}/p$ but the later factors need not be prime powers. We repeatedly use a certain divisibility trick to show that any factor which is not a prime power is necessarily an independent coordinate, thus reducing to the case handled in Section 3.1.

Finally, we prove Theorem 8 by analyzing the projection $\pi_0(H)$ to the first coordinate; the previous cases combine to show it must be either trivial (in which case H is contained in Z_0) or be equal to $\frac{n_0}{2}\mathbb{Z}/n_0\mathbb{Z}$, which allows us to reduce to the already-handled case $n_0 = 2$.

3.1. The case of p prime. We begin with the case where A is a finite abelian p -group with first factor $\mathbb{Z}/p$.

Proposition 9. *Let $A = \mathbb{Z}/p \times \prod_{i=1}^m \mathbb{Z}/p^{j_i}$, where p is prime and each $j_i \geq 1$. If H is a $\vec{z}_0$ -hyperplane in A , then H is a nearly- Z_0 hyperplane Z_Φ .*

Proof. We prove this by induction on m . The only $\vec{z}_0$ -hyperplane in $\mathcal{Z}(\mathbb{Z}/p)$ is $\{0\}$ itself, so the base case $m = 0$ is trivial.

Suppose H is a $\vec{z}_0$ -hyperplane of A . Write $A' = \mathbb{Z}/p \times \prod_{i=1}^{m-1} \mathbb{Z}/p^{j_i}$, so that $A = A' \times \mathbb{Z}/p^{j_m}$. Notice that

$$H_1 = \{\vec{x} \in A' \mid (\vec{x}, 1) \in H\}$$

is a $\vec{z}_0$ -subgroup of A' ; because there is an embedding $A'/H'_1 \hookrightarrow A/H'$ and A/H' is cyclic, so is A'/H'_1 , and thus H_1 is again a $\vec{z}_0$ -hyperplane. By our inductive hypothesis, we have $H_1 = Z_\phi$ for some homomorphism $\phi : \prod_{i=1}^{m-1} \mathbb{Z}/p^{j_i} \rightarrow \mathbb{Z}/2$. Our goal is to extend ϕ to a homomorphism $\Phi : \prod_{i=1}^m \mathbb{Z}/p^{j_i} \rightarrow \mathbb{Z}/2$ and verify that $H = Z_\Phi$.

As a first step on the way, we claim that the projection to the last factor is $\pi_m(H) = \mathbb{Z}/p^{j_m}$. Because $\pi_m(H)$ is an affine subgroup of $\mathbb{Z}/p^{j_m}$ which contains 1 (as $\pi_m(\vec{z}_0) = 1$), it takes the form $\pi_m(H) = 1 + p^r \mathbb{Z}/p^{j_m}$ for some $1 \leq p^r \leq p^{j_m}$. That $\pi_m(H)$ takes this form means there is some element $(b_0, \vec{b}, 1 + p^r) \in H$. Let $s_0 = b_0 - \phi(\vec{b}) - \phi(\vec{1}^{m-1})$; by taking affine combinations we see that

$$(s_0, \vec{1}^{m-1}, 1 + p^r) = (b_0, \vec{b}, 1 + p^r) - (\phi(\vec{b}) + \phi(\vec{1}^{m-1}), \vec{b}, 1) + (0, \vec{1}^{m-1}, 1) \in H.$$

Because $H \subset \mathcal{Z}(A)$, at least one coordinate must be zero; we divide into cases based on whether s_0 or $1 + p^r$ is zero.

Case 1: Suppose $s_0 = 0$; then $b_0 = \phi(\vec{b}) + \phi(\vec{1}^{m-1})$. By taking affine combinations we see that for all integers t and all $\vec{a} \in \prod_{i=1}^{m-1} \mathbb{Z}/n_i$, we have $(\phi(\vec{a}) + \phi(\vec{1}^{m-1}), \vec{a}, 1 + tp^r)$ equal to the affine combination

$$t(\phi(\vec{b}) + \phi(\vec{1}^{m-1}), \vec{b}, 1 + p^r) - t(\phi(\vec{b}) + \phi(\vec{1}^{m-1}), \vec{b}, 1) + (\phi(\vec{a}) + \phi(\vec{1}^{m-1}), \vec{a}, 1) \in H.$$

This establishes a containment $H_1 \times (1 + p^r \mathbb{Z}/p^{j_m}) \subset H$. Because $\pi_m(H')$ is equal to $p^r \mathbb{Z}/p^{j_m}$ and the kernel of π_m is given by H'_1 , comparing cardinalities shows that this containment is an equality. Now

$$A/H' \cong A'/H'_1 \times \mathbb{Z}/p^r \cong \mathbb{Z}/p \times \mathbb{Z}/p^r;$$

because A/H' is cyclic, we see that $p^r = 1$ and that $H = H_1 \times \mathbb{Z}/p^{j_m}$. Because $H_1 = Z_\phi$, the hyperplane H is equal to Z_Φ , where $\Phi(\vec{x}, i) = \phi(\vec{x})$.

Case 2: Suppose $s_0 \neq 0$, so that $1 + p^r = 0$. For this to be the case, we must have $p = 2$, $r = 0$, and $j_m = 1$. Additionally, s_0 must be 1, so $(1, \vec{1}^{m-1}, 0) \in H$. Write $\Phi : \prod_{i=1}^{m-1} \mathbb{Z}/2^{j_i} \times \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$ for the function defined by $\Phi(\vec{x}, i) = \phi(\vec{x}) + i$; we claim that

$$\begin{aligned} H = Z_\Phi &= \{(x_0, \dots, x_m) \mid x_0 = \Phi(x_1, \dots, x_m) + \Phi(\vec{1}^m)\} \\ &= \{(x_0, \dots, x_m) \mid x_0 = \phi(x_1, \dots, x_{m-1}) + \phi(\vec{1}^{m-1}) + x_m + 1\}. \end{aligned}$$

This amounts to the claim that the elements of H take one of the two forms

$$(\phi(\vec{x}) + \phi(\vec{1}^{m-1}), \vec{x}, 1) \quad \text{or} \quad (\phi(\vec{x}) + \phi(\vec{1}^{m-1}) + 1, \vec{x}, 0).$$

Because $H_1 = Z_\phi$ we know that $(\phi(\vec{x}) + \phi(\vec{1}^{m-1}), \vec{x}, 1) \in H$; taking affine combinations we see that

$$(\phi(\vec{x}) + \phi(\vec{1}^{m-1}) + 1, \vec{x}, 0) = (\phi(\vec{x}) + \phi(\vec{1}^{m-1}), \vec{x}, 1) + (1, \vec{1}^{m-1}, 0) - \vec{z}_0 \in H.$$

Thus $Z_\Phi \subset H \subset \mathcal{Z}(A)$. By Lemma 5, Z_Φ is maximal among hyperplanes contained in $\mathcal{Z}(A)$, so in fact we have $H = Z_\Phi$ as desired. $\square$

3.2. The general case. We begin by proving a brief technical lemma which illustrates a divisibility trick used twice in this section.

Lemma 10. *Let $A = \mathbb{Z}/4 \times A'$, where $A' = \prod_{i=1}^m \mathbb{Z}/n_i$ is a product of nontrivial cyclic groups. Suppose S is a $\vec{z}_0$ -subgroup. Then the projection to the initial coordinate $\pi_0(S)$ is not equal to $\mathbb{Z}/4$.*

Proof. Towards a contradiction, suppose $\pi_0(S) = \mathbb{Z}/4$, so that in particular $1 \in \pi_0(S)$. Thus there exists some $(1, \vec{a}) \in S$ for some $\vec{a} \in A$. By taking affine combinations, we see that

$$(k, k(\vec{a} - \vec{1}^m) + \vec{1}^m) = k(1, \vec{a}) + (1 - k)(0, \vec{1}^m) \in S$$

for each integer k . Because $S \subset \mathcal{Z}(A)$, we know one coordinate must be 0; in particular, for $k \not\equiv 0 \pmod{n_0}$ we have $k(1 - a_i) = 1$ for some $i > 0$, and hence k is invertible mod n_i . But if we take k to be the product of all primes dividing $|A|$, then $k \equiv 2 \not\equiv 0 \pmod{4}$, so every prime dividing $|A|$ (and hence every prime dividing n_i) is invertible mod n_i . This implies that $n_i = 1$, contradicting our assumption that the cyclic factors are nontrivial. $\square$

We now move to the case where A is a product of cyclic groups with first factor $\mathbb{Z}/p$. The central idea here is to use the divisibility trick to reduce the problem to the p -group case.

Proposition 11. *Let $A = \mathbb{Z}/p \times A'$, where $A' = \prod_{i=1}^m \mathbb{Z}/n_i$ is a product of finite cyclic groups. If H is a $\vec{z}_0$ -hyperplane, then H is contained in a nearly- Z_0 hyperplane Z_Φ .*

Proof. We begin by permuting the coordinates so that n_i is a power of p if and only if $0 \leq i \leq r$; we write $A = \mathbb{Z}/p \times A_p \times A_p^\perp$, where $A_p = \prod_{i=1}^r \mathbb{Z}/n_i$ and $A_p^\perp = \prod_{i=r+1}^m \mathbb{Z}/n_i$.

Consider the $\vec{z}_0$ -hyperplane $H_{\vec{1}} = \{\vec{x} \in \mathbb{Z}/p \times A_p \mid (\vec{x}, \vec{1}^{m-r}) \in H\}$ in A_p . By Proposition 9, we have $H_{\vec{1}} = Z_\phi$ for some homomorphism $\phi : A_p \rightarrow \mathbb{Z}/2$ supported on $\mathbb{Z}/2$ coordinates. Writing $\Phi : A_p \times A_p^\perp \rightarrow \mathbb{Z}/2$ for the map $\Phi(\vec{x}, \vec{y}) = \phi(\vec{x})$, we claim $H \subset Z_\Phi$.

Towards a contradiction, suppose not. Then there exists some $(y_0, \vec{y}, \vec{w}) \in H \setminus Z_\Phi$, which means that $y_0 \neq \phi(\vec{y}) + \phi(\vec{1}^r)$. We know that $(y'_0, \vec{y}, \vec{1}^{m-r}) \in Z_\phi \times \vec{1}^{m-r} \subset H$, where $y'_0 = \phi(\vec{y}) + \phi(\vec{1}^r)$.

By taking affine combinations we see that

$$(k(y_0 - y'_0), \vec{1}, k(\vec{w} - \vec{1}^{m-r}) + \vec{1}^{m-r}) = k(y_0, \vec{y}, \vec{w}) - k(y'_0, \vec{y}, \vec{1}^{m-r}) + \vec{z}_0 \in H \subset \mathcal{Z}(A).$$

Because $y_0 - y'_0$ is nonzero, for any $k \not\equiv 0 \pmod p$ the first coordinate is nonzero, so one of the final $m - r$ coordinates must be zero. This means that whenever $k \not\equiv 0 \pmod p$, we have $k(1 - w_i) \equiv 1 \pmod{n_i}$ for some $r < i \leq m$, and in particular k is invertible mod n_i . If k is the product of all primes other than p dividing $|A|$, this implies that n_i is a power of p ; this contradicts the assumption that n_i is only a power of p for $1 \leq i \leq r$. Thus $H \subset Z_\Phi$, as initially claimed. $\square$

We are now ready to prove Theorem 8. The argument amounts to reducing the first factor to $\mathbb{Z}/p$ and applying Proposition 11; the reduction step accounts for the potential for inflation in the initial coordinate of a nearly- Z_0 hyperplane.

Proof of Theorem 8. Recall that $A = \prod_{i=0}^m \mathbb{Z}/n_i$ and H is a $\vec{z}_0$ -hyperplane in A . We know that $\pi_0(H)$ is an affine subgroup of $\mathbb{Z}/n_0$ containing 0, so it must be a linear subgroup $r\mathbb{Z}/n_0$ for some $r \mid n_0$. If $r = 0$, then we have $H \subset Z_0$, so we may assume that $\pi_0(H)$ is nontrivial (and hence $0 < r < n_0$).

Let $p \mid n_0/k$ with p prime or $p = 4$. Define

$$\bar{H} = \{(x_0, \vec{x}) \mid (n_0 x_0/p, \vec{x}) \in H\} \subset \mathbb{Z}/p \times \prod_{i=1}^m \mathbb{Z}/n_i = \bar{A}.$$

Now $\bar{H}$ is a $\vec{z}_0$ -hyperplane in $\bar{A}$ with $\pi_0(\bar{H}) = \mathbb{Z}/p$ (in particular, $\bar{H}$ is not contained in Z_0). By Lemma 10 we cannot have $p = 4$, so p is prime; by Proposition 11, we see that $\bar{H} \subset Z_\phi$ for a **nontrivial** homomorphism $\phi : \prod_{i=1}^m \mathbb{Z}/n_i \rightarrow \mathbb{Z}/2$. In particular, $p = 2$.

Thus neither 4 nor an odd prime divides n_0/r , so $n_0 = 2r$ and $\pi_0(S) = r\mathbb{Z}/2r$. This implies that H is contained in the corresponding inflated hyperplane $Z_\phi \subset A$, the inflation of $Z_\phi \subset \bar{A}$ by r in the initial coordinate. $\square$

4. PROOFS OF THEOREMS 1 AND 2

Using our results about $\vec{z}_i$ -hyperplanes, we have the following strengthening of Theorem 1 from the introduction. Because we no longer focus on a particular special coordinate, we return to the convention from the introduction that indexing of products begins at $i = 1$.

Theorem 12. *If $\mathcal{H} = \{H_1, \dots, H_{m'}\}$ is a hyperplane splitting of $A = \prod_{i=1}^m \mathbb{Z}/n_i$ with $U(\mathcal{H}) = \mathcal{Z}(A)$, then up to a permutation of the H_i we have*

- (a) *When $n_i > 2$, we have $\vec{z}_i \in H_i$;*
- (b) *Each H_i is a nearly-coordinate hyperplane;*
- (c) *$m = m'$ and each A/H'_i is a cyclic group of order n_i .*

Proof. For convenience, we suppose (by permuting the n_i) that $n_i > 2$ if and only if $1 \leq i \leq \ell$. We begin by handling the hyperplanes whose quotients have order larger than 2.

Because $U(\mathcal{H}) = \mathcal{Z}(A)$ and each $\vec{z}_i$ lies in $\mathcal{Z}(A)$, we see that there exists some $H_{j(i)}$ so that $\vec{z}_i \in H_{j(i)}$. For $1 \leq i \leq \ell$, we know from Lemma 7 that if some H_j contains $\vec{z}_i$, it contains no other $\vec{z}_{i'}$; for $1 \leq i \leq \ell$, we see that if $j(i) = j(i')$, we must have $i = i'$. Thus we may permute the H_i so that $j(i) = i$ for all $1 \leq i \leq \ell$, so that $\vec{z}_i \in H_i$ for all i with $n_i > 2$. This establishes (a).

Next, observe that when $1 \leq i \leq \ell$, each H_i is a **maximal** $\vec{z}_i$ -hyperplane by Lemma 4 and the assumption $U(\mathcal{H}) = \mathcal{Z}(A)$. By our classification result Theorem 8, we see that when $1 \leq i \leq \ell$ each H_i is nearly Z_i (and hence has order n_i), which establishes (b) and (c) for $1 \leq i \leq \ell$.

We conclude by showing that all remaining hyperplanes have order 2 and that there are $m - \ell$ of them, which immediately gives (c); the rest of (b) then follows from Lemma 6.

By definition of hyperplane splitting, we know that

$$\pi : A \rightarrow \prod_{j=1}^{m'} A/H'_j$$

is an isomorphism, so the corresponding homomorphism

$$\bigcap_{i=1}^{\ell} H'_i \rightarrow \prod_{j=\ell+1}^{m'} A/H'_j$$

is also an isomorphism. Now we may identify the domain with $(\mathbb{Z}/2)^{m-\ell}$ as follows. For $1 \leq i \leq \ell$ each H_i is nearly Z_i , so H'_i takes the form

$$H'_i = \{(x_1, \dots, x_m) \in A \mid x_i = \phi_i(x_{\ell+1}, \dots, x_m)\},$$

where $\phi_i : (\mathbb{Z}/2)^{m-\ell} \rightarrow \mathbb{Z}/n_i$ is a homomorphism. Then

$$\bigcap_{i=1}^{\ell} H'_i = \{(x_1, \dots, x_m) \in A \mid \forall 1 \leq i \leq \ell \quad x_i = \phi_i(x_{\ell+1}, \dots, x_m)\}.$$

Thus the projection to the last $m - \ell$ coordinates

$$\bigcap_{i=1}^{\ell} H'_i \rightarrow \prod_{j=\ell+1}^m \mathbb{Z}/2$$

is an isomorphism: the first ℓ coordinates are uniquely determined by the last $m - \ell$, which are themselves unconstrained.

We have proved that $\prod_{j=\ell+1}^{m'} A/H'_j$ is isomorphic to $(\mathbb{Z}/2)^{m-\ell}$. Because each A/H'_j is cyclic and isomorphic to a subgroup of $(\mathbb{Z}/2)^{m-\ell}$, we see that each A/H'_j is cyclic of order 2. By comparing cardinalities, we see that $m - \ell = m' - \ell$, as desired. $\square$

Notice that when $n_i > 2$ for all i , nearly-coordinate hyperplanes are coordinate, in which case this statement asserts that there is a unique hyperplane splitting (the coordinate hyperplane splitting) with union equal to $\mathcal{Z}(A)$. This proves Theorem 1.

4.1. Pontryagin duals and Theorem 2. To give the proof of Theorem 2, we will want to pass from information about the behavior of hyperplanes in A to information about the behavior of the basis vectors of A . The *Pontryagin dual* $A^\vee = \text{Hom}(A, \mathbb{R}/\mathbb{Z})$ is the right tool to do so. For a general finite abelian group A , the groups A and $A^\vee$ are noncanonically isomorphic. When $A = \prod_{i=1}^m \mathbb{Z}/n_i$ is a product of finite cyclic groups, there is a canonical isomorphism

$$\prod_{i=1}^m \mathbb{Z}/n_i \cong A^\vee$$

given by sending $\vec{x} = (x_1, \dots, x_m)$ to the homomorphism $\varphi_{\vec{x}}$, defined by

$$\varphi_{\vec{x}}(a_1, \dots, a_m) = \sum_{i=1}^m \frac{a_i x_i}{n_i}.$$

We freely use this isomorphism in what follows. With respect to this isomorphism, the j 'th basis vector of $A^\vee$ is given by the homomorphism

$$e_j^\vee(a_1, \dots, a_m) = \frac{a_j}{n_j}.$$

It will be convenient to suppose the factors are ordered so that

$$A = (\mathbb{Z}/2)^\ell \times \prod_{i=\ell+1}^m \mathbb{Z}/n_i,$$

where $n_i > 2$ for $i > \ell$. We will write $A_2 = (\mathbb{Z}/2)^\ell \times \{0\}$. Again, $A^\vee$ naturally splits as $A_2^\vee \times \prod_{i=\ell+1}^m \mathbb{Z}/n_i$.

Lemma 13. *The Pontryagin dual group has the following properties.*

(a) *An affine hyperplane $H \subset A$ determines a (linear) cyclic subgroup*

$$C_H = \{\phi \in A^\vee \mid \phi(H') = 0\}$$

with $|C_H| = |A/H'|$.

(b) *If $f : A \rightarrow B$ is a homomorphism, then so is the map $f^\vee : B^\vee \rightarrow A^\vee$, defined by $f^\vee(\phi) = \phi f$. If $H \subset B$ is a hyperplane and $J = f^{-1}(H)$, then $f^\vee(C_H) = C_J$.*

(c) *If $A = \prod_{i=1}^m \mathbb{Z}/n_i$ and $B = \prod_{j=1}^{m'} \mathbb{Z}/n'_j$, then f can be represented by a matrix (f_{ij}) .*

The matrix representation of $f^\vee$ is given by $f_{ij}^\vee = \frac{n_j f_{ji}}{n'_i}$.

(d) *If H is a nearly- Z_i hyperplane, then $C_H = \langle e_i^\vee + \phi^\vee \rangle$, where $\phi^\vee \in A_2^\vee$.*

We omit the proof of these claims, which are straightforward exercises. These facts in hand, we are ready to complete the proof of Theorem 2.

Proof of Theorem 2. Suppose

$$f : A = (\mathbb{Z}/2)^\ell \times \prod_{i=\ell+1}^m \mathbb{Z}/n_i \rightarrow (\mathbb{Z}/2)^{\ell'} \times \prod_{j=\ell'+1}^{m'} \mathbb{Z}/n'_j = A'$$

is an isomorphism with $f(\mathcal{Z}(A)) = \mathcal{Z}(A')$, and for which $n_i, n'_j > 2$ for all $i > \ell$ and $j > \ell'$.

Write $\mathcal{H}_A = \{W_1, \dots, W_m\}$ and $\mathcal{H}_{A'} = \{Z_1, \dots, Z_{m'}\}$ for the respective coordinate hyperplane splittings. Then f sends $\mathcal{H}_A$ to a hyperplane splitting $f(\mathcal{H}_A)$ with

$$U(f(\mathcal{H}_A)) = f(\mathcal{Z}(A)) = \mathcal{Z}(A').$$

Write $f(W_i) = \langle \phi_i^\vee \rangle$ for some $\phi_i \in A^\vee$. Because f is an isomorphism, we have $f^{-1}(f(W_i)) = W_i$, so by Lemma 13(b) we see that $f^\vee e_i^\vee = c_i \phi_i^\vee$ for some scalar c_i . We need to compute $\phi_i^\vee$.

Applying Theorem 12, we see that $\ell = \ell'$ and $m = m'$. Each of these are nearly-coordinate; up to a permutation Theorem 12(a) asserts that for $\ell < i \leq m$, each $f(W_i)$ is a $\tilde{z}_i$ -hyperplane. By Lemma 7 these combine to show that when $\ell < i \leq m$ each $f(W_i)$ is nearly- Z_i . Then Lemma 13(d) implies that $\phi_i^\vee = e_i^\vee + \psi_i^\vee$ for some $\psi_i^\vee \in A_2^\vee$ when $\ell < i \leq m$.

This also holds for $1 \leq i \leq \ell$, though the argument is more roundabout: we know that $f(W_i)$ is nearly-coordinate by Lemma 6, so $\phi_i^\vee$ is of the form $c_i(e_j^\vee + \eta_i^\vee)$ for some $e_j^\vee$ (possibly with $i \neq j$) and some $\eta_i^\vee \in A_2^\vee$. Because $e_i^\vee - e_j^\vee$ lies in $A_2^\vee$, so we may simply rewrite this as $c_i(e_i^\vee + \psi_i^\vee)$, taking $\psi_i^\vee = \eta_i^\vee + (e_i^\vee - e_j^\vee)$.

A matrix representation of $f^\vee$ is thus given by $f^\vee = C + M_2$, where C is diagonal with entries $C_{ii} = c_i$ and M_2 is supported in the first ℓ rows (its i 'th column is $c_i \psi_i^\vee \in A_2^\vee$). Therefore $f^\vee$ has matrix form

$$f^\vee = \begin{pmatrix} f_{11}^\vee & f_{12}^\vee \\ 0 & f_{22}^\vee \end{pmatrix},$$

where $f_{22}^\vee$ is diagonal; because $f^\vee$ is an isomorphism and this matrix is block-diagonal, $f_{11}^\vee$ must be an isomorphism.

By applying Lemma 13(c), we see that the matrix representation of f is a weighted transpose of the matrix representation of $f^\vee$, from which we see that

$$f = \begin{pmatrix} f_{11} & 0 \\ f_{21} & D \end{pmatrix},$$

where f_{11} is invertible and $f_{22} = D$ is diagonal. □

Remark 5. In fact, because Theorem 12 allows for affine hyperplanes (and affine hyperplane splittings), Theorem 2 can be slightly strengthened: the map $f(x) = g(x) + c$ can be taken to be an affine isomorphism (so g is an isomorphism and $c \in A'$ is some constant). The conclusion for g is the same, and the conclusion for c is $c \in A_2 \times \{0\}$.

5. AN APPLICATION TO TOPOLOGY

Given a compact oriented $2n$ -dimensional manifold X (possibly with boundary), consider the cohomology groups $H^n(X; \mathbb{C})$ and the relative cohomology groups $H^n(X, \partial X; \mathbb{C})$; we suppress the complex coefficients when clear from context. There is a natural map $H^n(X, \partial X) \rightarrow H^n(X)$, and intermediate to these lies

$$\widehat{H}^n(X) = \text{image}(H^n(X, \partial X) \rightarrow H^n(X)).$$

Lefschetz duality [Hat02, Theorem 3.43] asserts that the cup product and pairing with the fundamental class defines nondegenerate bilinear pairings

$$H^n(X, \partial X) \otimes H^n(X) \rightarrow \mathbb{C} \leftarrow H^n(X) \otimes H^n(X, \partial X),$$

with the pairings satisfying $B_X(x, y) = (-1)^n B'_X(y, x)$. It is straightforward to verify that nondegeneracy of B_X implies the induced pairing

$$\widehat{B}_X : \widehat{H}^n(X) \otimes \widehat{H}^n(X) \rightarrow \mathbb{C}$$

is nondegenerate.

When X is $4k$ -dimensional, so that $n = 2k$ and $\widehat{B}_X$ is a nondegenerate symmetric bilinear pairing, we say that its *signature* is $\sigma(X) = \sigma(\widehat{B}_X)$, the number of positive eigenvalues of $\widehat{B}_X$ minus the number of negative eigenvalues (both counted with multiplicity).

When X is not simply connected, one may take (co)homology with *local coefficients* [Hat02, Section 3.H]. In particular, given an element $\phi \in \text{Hom}(\pi_1 X, S^1)$, we may construct a complex line bundle $\mathbb{C}_\phi$ over X with monodromy ϕ . There is again a nondegenerate pairing $H^n(X, \partial X; \mathbb{C}_\phi) \otimes H^n(X; \mathbb{C}_\phi) \rightarrow \mathbb{C}$, now given by the composite

$$H^n(X, \partial X; \mathbb{C}_\phi) \otimes H^n(X; \mathbb{C}_\phi) \rightarrow H^{2n}(X, \partial X; \mathbb{C}_\phi \otimes \mathbb{C}_\phi) \xrightarrow{(z_1 \otimes z_2)x \mapsto \langle z_1, z_2 \rangle x} H^{2n}(X, \partial X; \mathbb{C}) \cong \mathbb{C}.$$

Once again we consider the intermediate middle cohomology groups $\widehat{H}^n(X; \mathbb{C}_\phi)$, and the restricted pairing $\widehat{B}_{X, \phi}$ is nondegenerate on $\widehat{H}^n(X; \mathbb{C}_\phi)$. This is discussed in [APS75, Section 2], where one finds the following definition of twisted signatures.

Definition 6. *If X is a compact oriented $4k$ -dimensional manifold, its **twisted signature** with respect to $\phi \in \text{Hom}(\pi_1 X, S^1)$ is*

$$\sigma(X, \phi) = \sigma(\widehat{B}_{X, \phi}).$$

We say that X is **signature-simple** if X is connected and $\sigma(X, \phi) = 0$ if and only if ϕ is the trivial homomorphism.

We have the following basic properties of the twisted signatures.

Lemma 14. *Twisted signatures of $4k$ -dimensional manifolds (X, ϕ) have the following properties:*

- (a) *If $f : X \rightarrow Y$ is a homeomorphism, then $\sigma(X, f^* \psi) = \sigma(Y, \psi)$.*
- (b) *If X and Y are of the same dimension, then*

$$\sigma(X \# Y, \phi \# \psi) = \sigma(X, \phi) + \sigma(Y, \psi).$$

- (c) *If $\phi \in \text{Hom}(H_1 X, S^1)$ and $\psi \in \text{Hom}(H_1 Y, S^1)$, where X and Y are compact oriented manifolds of dimension divisible by 4, we have*

$$\sigma(X \times Y, \phi \cdot \psi) = \sigma(X, \phi) \cdot \sigma(Y, \psi).$$

- (d) *If X is a compact oriented 4-manifold, then $\sigma(X, \phi) - \sigma(X) = -\rho(\partial X, \phi|_{\partial X})$, where $\rho(Y, \alpha)$ is the topological invariant of [APS75].*

Proof. The first statement follows immediately from naturality of the cohomology groups and the cup product; f induces an isomorphism between $\widehat{H}^{2k}(Y; \mathbb{C}_\psi)$ and $\widehat{H}^{2k}(X; \mathbb{C}_{f^* \psi})$ which sends $\widehat{B}_{Y, \psi}$ to $\widehat{B}_{X, f^* \psi}$, and hence they have the same signature.

The second statement follows from the fact that there is a direct sum decomposition $\widehat{H}^{2k}(X \# Y; \mathbb{C}_{\phi \# \psi}) \cong \widehat{H}^{2k}(X; \mathbb{C}_\phi) \oplus \widehat{H}^{2k}(Y; \mathbb{C}_\psi)$ with respect to which $\widehat{B}_{X \# Y, \phi \# \psi}$ is identified with $\widehat{B}_{X, \phi} \oplus \widehat{B}_{Y, \psi}$.

The third statement is well-known for untwisted signatures: it follows from a combination of the Künneth formula $H^{2k+2\ell}(X \times Y; \mathbb{C}) \cong \bigoplus_{i+j=k+\ell} H^{2i}(X; \mathbb{C}) \otimes H^{2j}(Y; \mathbb{C})$ and a linear algebra lemma [CHS57, Lemma 1]. Because the homomorphism $H_1(X \times Y) \rightarrow S^1$ factors as a product of $\phi : H_1(X) \rightarrow S^1$ and $\psi : H_1(Y) \rightarrow S^1$, an appropriate Künneth formula holds for local coefficients, and the linear algebra fact applies without change to show the desired multiplicativity property.

The final statement is a corollary of the Atiyah–Patodi–Singer index theorem, explicitly stated as [APS75, Theorem 2.4]. $\square$

Remark 6. The final statement implies that the property of being ‘signature-simple’ is equivalent to the claim that $\sigma(X) = 0$ and $\rho(\partial X; i^* \phi) \neq 0$ for all $\phi \neq 1$, and hence is almost entirely a property of the boundary. In particular, when X is a closed manifold, the value of $\sigma(X, \phi)$ is independent of ϕ — and thus X can never be signature-simple.

This in hand, we can prove Theorem 3. After giving the proof, we give a construction of an infinite family of signature-simple manifolds.

Proof of Theorem 3. Let $X(n)$ be the family of signature-simple manifolds, and Y a manifold with $\sigma(Y) = 1$ and $H_1(Y) = 0$. Let $X = \prod_{i=1}^m X(n_i)$ and $X' = \prod_{i=1}^{m'} X(n'_i) \times Y$, and suppose $f : X \rightarrow X'$ is a homeomorphism.

Observe that $H_1(X) \cong \prod_{i=1}^m \mathbb{Z}/n_i$, and this induces an isomorphism

$$\mathrm{Hom}(\pi_1 X, S^1) \cong \mathrm{Hom}(H_1 X, S^1) \cong \prod_{i=1}^m \mathbb{Z}/n_i,$$

as discussed before Lemma 13. With respect to this isomorphism, we have

$$\sigma(X, \phi_1, \dots, \phi_m) = \prod_{i=1}^m \sigma(X(n_i), \phi_i)$$

by Lemma 14(c). By the assumption that the $X(n_i)$ are signature-simple, we see that $\sigma(X, \phi_1, \dots, \phi_m) = 0$ if and only if some ϕ_i is trivial. Thus $\sigma(X)^{-1}(0) = \mathcal{Z}(A')$ is the union of coordinate hyperplanes in $A' = \mathrm{Hom}(H_1 X, S^1) \cong \prod_{i=1}^m \mathbb{Z}/n_i$.

Similarly, $H_1(X') \cong \prod_{j=1}^{m'} \mathbb{Z}/n'_j$, with no factor from Y because $\tilde{H}_0(Y) = H_1(Y) = 0$. With respect to this decomposition, we have

$$\sigma(X', \psi_1, \dots, \psi_{m'}) = \sigma(Y) \prod_{j=1}^{m'} \sigma(X(n'_j), \psi_j).$$

Because $\sigma(Y) \neq 0$ and the $X(n'_j)$ are signature-simple, we see once again that $\sigma(X')^{-1}(0) = \mathcal{Z}(A)$ is the union of coordinate hyperplanes in $A = \mathrm{Hom}(H_1(X'), S^1) \cong \prod_{j=1}^{m'} \mathbb{Z}/n'_j$.

By Lemma 14(a), we see that $\sigma(X, f^* \psi) = \sigma(X', \psi)$. In particular, f^* sends $\sigma(X')^{-1}(0) = \mathcal{Z}(A)$ to $\sigma(X)^{-1}(0) = \mathcal{Z}(A')$. Because $n_i > 2$ for all i (as the n_i are all odd and larger than 1), Theorem 2 applies; we see that $m = m'$ and after a permutation of the n'_j we have $n_i = n'_i$ and f^* diagonal. It follows that the pushforward map f_* on homology — being the transpose of f^* with respect to the given direct-sum decomposition — is also diagonal.

To complete the argument, we should explain why Y is a singleton. But homeomorphisms of topological manifolds preserve their dimensions [Hat02, Theorem 2B.3], and $\dim X = 4m$ while $\dim X' = 4m' + \dim Y$. We have established that $m = m'$, so $\dim X = \dim X'$ implies that $\dim Y = 0$, and hence Y is a disjoint union of points. Because X is connected, so too is X' , and hence so is Y ; so Y is a singleton, as desired. $\square$

We conclude by explaining how to construct a family of signature-simple manifolds (albeit by an inexplicit construction).

Definition 7. Let $n > 1$ be any integer and $0 < q < n$ be coprime to n . We say a compact connected oriented 4-manifold X is a **signature** $L(n, q)$ if there is an isomorphism $r : \text{Hom}(\pi_1 X, S^1) \cong \text{Hom}(\pi_1 L(n, q), S^1)$ inducing an equality

$$\sigma(X, \varphi) = -n\rho(L(n, q), r(\varphi)).$$

Lemma 15. *There exists a signature $L(n, q)$ for all (n, q) .*

Proof. We will construct the desired manifold in three steps.

- (1) The first step is the most complicated: we construct a 4-manifold X_0 with boundary $\partial X_0 \cong \sqcup_n L(n, q)$ and so that the homomorphism $\phi : H_1(L(n, q)) \rightarrow \mathbb{Z}/n$ defined by the composite $H_1(\sqcup_n L(n, q); \mathbb{Z}) \cong (\mathbb{Z}/n)^n \xrightarrow{x \mapsto \sum x_i} \mathbb{Z}/n$ extends to a (necessarily surjective) homomorphism $\Phi_0 : H_1(X_0) \rightarrow \mathbb{Z}/n$. As we now explain, this is equivalent to the claim that $n[L(n, q)]$ represents the zero element in a certain group $\Omega_3^{SO}(B\mathbb{Z}/n)$.

For any group G one may define a *classifying space* BG [Hat02, Section 1.B] with the property that homotopy classes of maps $X \rightarrow BG$ correspond bijectively to homomorphisms $\pi_1(X) \rightarrow G$ up to conjugacy. Further, given any space Y , one may associate its oriented bordism group $\Omega_d^{SO}(Y)$, whose elements are equivalence classes of pairs (M, f) , where M is a closed oriented d -manifold and $f : M \rightarrow Y$ is a continuous map; we consider (M, f) equivalent to 0 if there is a compact oriented $(d+1)$ -manifold W with $\partial W = M$ and an extension of f to $F : W \rightarrow Y$. Addition in this group is given by disjoint union.

Combining these, the closed oriented 3-manifold $L(n, q)$ comes equipped with a map $\pi_1 L(n, q) \rightarrow \mathbb{Z}/n$, hence defines a class $[L(n, q)] \in \Omega_3^{SO}(B\mathbb{Z}/n)$, and $n[L(n, q)]$ is represented by the disjoint union of n copies of $L(n, q)$. We claim above that $\sqcup_n L(n, q)$ bounds a manifold W equipped with a map $\pi_1(X_0) \rightarrow \mathbb{Z}/n$ (and because $\mathbb{Z}/n$ is abelian, this homomorphism factors through $H_1(X_0)$), and this is equivalent to the claim that $n[L(n, q)] = 0 \in \Omega_3^{SO}(B\mathbb{Z}/n)$. To conclude, it suffices to observe that $\Omega_3^{SO}(B\mathbb{Z}/n) \cong \mathbb{Z}/n$, as follows for instance from Sections 7 and 15 of [CF64].

- (2) Next we construct a manifold X_1 with the same property as in the previous example but so that $\Phi_1 : H_1(X_1) \rightarrow \mathbb{Z}/n$ is an isomorphism. To do so, pick a finite set of disjoint embedded loops $\gamma_1, \dots, \gamma_k$ whose homology classes generate $\ker(\Phi_0)$. Perform surgery on each: delete a tubular neighborhood of each diffeomorphic to $S^1 \times D^3$, and glue in a copy of $D^2 \times S^2$ on each resulting boundary component. A Mayer–Vietoris argument shows that $H_1(X_1)$ can be identified with the quotient of $H_1(X_0)$ by the subgroup generated by the $[\gamma_i]$ — that is, $\ker(\Phi_0)$ — and hence $\Phi_1 : H_1(X_1) \rightarrow \mathbb{Z}/n$ is an isomorphism. In particular, we may take $\phi \mapsto \phi|_{\partial X}$ to be our isomorphism r .
- (3) Finally, by taking the connected sum of X_1 with copies of $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$ as appropriate — which have signature ± 1 and trivial first homology — we may obtain a manifold X with the properties above and which also has $\sigma(X) = 0$.

Now the isomorphism $\partial X \cong \sqcup_n L(n, q)$ is chosen so that the map $i : H_1(\partial X) \rightarrow H_1(X) \cong \mathbb{Z}/n$ is the coordinate-summing map on first homology, and hence the map

$$\mathbb{Z}/n \cong \text{Hom}(H_1 X, S^1) \xrightarrow{i^*} \text{Hom}(H_1(\sqcup_n L(n, q)), S^1) = \text{Hom}(H_1 L(n, q), S^1)^n \cong (\mathbb{Z}/n)^n$$

is the diagonal map. Finally, together with additivity of the ρ invariant under disjoint unions, it follows from Lemma 14(d) that

$$\sigma(X, \phi) = \sigma(X, \phi) - \sigma(X) \cong -\rho(\sqcup_n L(n, q); \phi|_{\partial X}) = -n\rho(L(n, q), \phi|_{\partial X}). \quad \square$$

Write $\rho_{n,q}(i) = \rho(L(n, q), \phi_i)$ for the function $\mathbb{Z}/n \rightarrow \mathbb{Q}$ determined by an isomorphism $\text{Hom}(H_1 L(n, q), S^1) \cong \mathbb{Z}/n$. The following lemma is valid for any choice of such isomorphism, though the computation we use is in terms of a specific one.

Lemma 16. *For all (n, q) coprime and n odd, we have $\rho_{n,q}(k) = 0$ if and only if $k \equiv 0 \pmod n$.*

Proof. Let n be arbitrary and let q be coprime to n . If $\rho_{n,q}(k) \equiv 0 \pmod{2\mathbb{Z}}$ for some $0 < k < n$, we will show that n is even. By [Tof19, Corollary 3.3.25], we have for $0 < k < n$

$$\rho_{n,q}(k) = 2 \left(\left(\frac{kq}{n} \right) \right) - 4 \sum_{j=1}^k \left(\left(\frac{jq}{n} \right) \right),$$

where $((x))$ is the sawtooth function

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases}$$

Notice that $((x))$ simplifies drastically modulo $\mathbb{Z}$, as the floor term disappears. Further notice that because q is prime to n and $0 < k < n$, all appearances of $((x))$ in $\rho_{n,q}(k)$ have x non-integer, so we may simply replace this modulo $\mathbb{Z}$ with $x - \frac{1}{2}$.

Take the expression $\rho_{n,q}(k)$ modulo $2\mathbb{Z}$. The whole expression simplifies to

$$\rho_{n,q}(k) \equiv \frac{2kq}{n} - 1 - 4 \frac{qk(k+1)}{2n} \equiv -1 - \frac{2k^2q}{n} \pmod{2\mathbb{Z}}.$$

If $\rho_{n,q}(k) \equiv 0 \pmod{2\mathbb{Z}}$, multiplying through by n we see that $0 \equiv 2k^2q \equiv -n \pmod{2\mathbb{Z}}$; this implies that n must be even. $\square$

Remark 7. It is straightforward but slightly more tedious to extend the proof of Lemma 16 to $n = 4^k m$ and m odd. Lemma 16 may still be true for $n = 2^{2k+1}m$, but the argument is necessarily more subtle: for instance, we have $\rho_{6,1}(3) = \rho_{8,1}(2) = 2 \equiv 0 \pmod{2\mathbb{Z}}$.

Finally, Lemmas 15 and 16 combine to show

Corollary 17. *For odd n , there exists a signature-simple 4-manifold X with $H_1(X) \cong \mathbb{Z}/n$.*

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