Heegaard Floer homology is not cup homology

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Abstract

The Heegaard Floer homology group $HF^\infty(Y;R)$ is known to depend only on the triple-cup product form of Y. A related group, the *cup homology* $HC^\infty(Y;R)$, is defined explicitly, often easy to compute, and enjoys a spectral sequence $HC^\infty \implies HF^\infty$. Ozsvath and Szabo conjectured that this spectral sequence collapses. An explicit counterexample shows that this is false, even over $R = \mathbb{F}_2$. This invalidates a preprint claiming a proof over $\mathbb{F}_2$, but otherwise has little effect on the published literature.

Introduction

Let $\omega \in \Lambda^3(\mathbb{Z}^n)$ be a 3-form. Fixing a commutative ring R, one may define the associated *cup complex* to be

$$C^\infty(\omega; R) = \left(\Lambda^*(\mathbb{Z}^n) \otimes_{\mathbb{Z}} R, d_\omega x = \omega \wedge x\right).$$

Its homology groups $HC^\infty(\omega; R)$ are called the *cup homology* of ω. If Y is a 3-manifold, the group $H^1(Y;\mathbb{Z})$ has a 3-form, the *triple-cup product*

$$\omega_Y(x, y, z) = \langle x \smile y \smile z, [Y]\rangle.$$

The *cup homology groups* of Y [Mar08] are defined to be $HC^\infty(\omega_Y; R)$. There is an anticipated relationship with Heegaard Floer homology:

Conjecture 1 ([OS03, Conjecture 4.10])**.** *There is a spectral sequence*

$$E^3 = C^\infty(\omega_Y) \otimes_{\mathbb{Z}} R \implies HF^\infty(Y;R).$$

This spectral sequence collapses on the E^4 page.

The first part of this conjecture is now known. Such a spectral sequence is constructed in [Lid13] and used to analyze HF^∞. A related spectral sequence in monopole Floer homology $\overline{HM}$ is established in [KM07, Chapter 35], and shown to collapse over $\mathbb{R}$. Monopole Floer homology is isomorphic to Heegaard Floer homology [KLT20, CGH24], so we may use either theory. Using the perspective of monopole Floer homology, Kronheimer and Mrowka prove that $\overline{HM}(Y)$ is always non-trivial, and that $\overline{HM}(Y) \cong \Lambda^*(\mathbb{Z}^n)$ if and only if $\omega_Y = 0$; the latter result was also obtained over $\mathbb{F}_2$ in [Lid13].

It follows from either [Lid13] or [KM07] that $HF^\infty(Y)$ depends only on the triple-cup product ω_Y; neither approach makes explicit the dependence or the structure of the higher differentials.

Francesco Lin and the author, using the perspective of Monopole Floer homology, showed that $\overline{HM}(Y)$ can be computed as the *extended cup homology* of the 3-form ω_Y [LME24]. Precisely, if x_3 is a 3-form, there exist certain odd forms $x_{2n+1} = x_3^{\circ n}$ so that $\overline{HM}(Y) \cong HC^\infty(\omega_Y + \omega_Y^{\circ 2} + \cdots)$ The higher forms x_{2n+1} can be computed explicitly in terms of the expression for ω_Y in a chosen basis for $H^1(Y)$. See [LME24, Section 5] for a precise definition. Thus, Conjecture 1 becomes a question in combinatorial algebra.

This leads to a few positive results. Using the description in terms of extended cup homology, it is not difficult to show that the spectral sequence collapses for $b_1(Y) \leqslant 7$. A few million computations gave no counterexample, even over $\mathbb{Z}$. Analisa Faulkner Valiente and the author also showed that the conjecture holds for $Y = S^1 \times \Sigma_g$, though there is no *natural* isomorphism between the cup homology and extended cup homology groups for $g = 3$.

Nevertheless, the conjecture is false:

Theorem 2. *If Y is a 3-manifold with*

$$\begin{aligned}\omega_Y = \;& e^1(e^{3,6} + e^{4,5} + e^{7,10} + e^{8,9} + e^{11,14} + e^{12,13}) \\ &+ e^2(e^{3,5} + e^{4,5} + e^{4,6} + e^{7,9} + e^{8,9} + e^{8,10} + e^{11,13} + e^{12,13} + e^{12,14}),\end{aligned}$$

then the E^5 page of the spectral sequence has nonzero differential, and

$$\dim HC^\infty(\omega_Y; \mathbb{F}_2) = 5184, \qquad \dim HF^\infty(Y; \mathbb{F}_2) = 5120.$$

It should be noted that this contradicts the main result of [Lid10], which claims the spectral sequence collapses over $\mathbb{F}_2$. While most arguments seem reasonable, there appears to be an error in the proof of Lemma 10.1, where the author observes that one has $d_3^2 = (d_3 - d_3^{K_i})^2 = 0$ and asserts that $(d_3^{K_i})^2 = 0$ as well. It is unclear how this conclusion is reached; it does not seem that d_3 and $d_3^{K_i}$ are known to commute.

This error has no significant effect on the literature. The most important facts about $HF^\infty(Y)$ are that it is non-trivial over $\mathbb{R}$ for all Y, and that it is standard precisely when $\omega_Y = 0$. These facts are not in doubt, having published proofs from both the perspective of Heegaard Floer homology and monopole Floer homology. The author was only able to identify one reference to [Lid10] which uses the main result in a substantive way: [HM18, Remark 2.7] uses it to give a lower bound on the τ-invariant of a Dehn surgery. The refined bound is not needed, and in any case may well remain true after a more careful investigation of the spectral sequence.

Proof. The E^{2n+2} page of the spectral sequence is isomorphic to $HC^\infty(\sum_{i=1}^n \omega_Y^{\circ i})$. It suffices to show the ranks of the maps $x \mapsto \sum_{i=1}^n x \wedge \omega_Y^{\circ i}$ disagree for $n = 1, 2$. (It is also the case that the ranks are the same for all $n \geqslant 2$.)

The definition of the insertion square $\omega_Y^{\circ 2}$ is fairly simple; higher powers are somewhat more intricate, so will be omitted from the discussion. Set $\iota(e^I, e^J) = 0$ unless $|I \cap J| = 1$, in which case write $J = J_1 \cup \{j\} \cup J_2$ with $J_1 < j < J_2$. Then $\iota(e^I, e^J) = e^{J_1} \wedge e^I \wedge e^{J_2}$. If $x_3 = \sum_I a_I e^I$, its insertion square is

$$x_3^{\circ 2} = \frac{1}{2} \sum_{I,J} a_I a_J \, \iota(e^I, e^J).$$

It is straightforward to compute that

$$\begin{aligned}
\omega_Y^{\circ 2} = \quad & e^1 \big(e^{3,4,5,6} + e^{3,6,7,10} + e^{3,6,8,9} + e^{3,6,11,14} + e^{3,6,12,13} \\
& \quad + e^{4,5,7,10} + e^{4,5,8,9} + e^{4,5,11,14} + e^{4,5,12,13} + e^{7,8,9,10} \\
& \quad + e^{7,10,11,14} + e^{7,10,12,13} + e^{8,9,11,14} + e^{8,9,12,13} + e^{11,12,13,14} \big) \\
+ \, & e^2 \big(e^{3,4,5,6} + e^{3,5,7,9} + e^{3,5,8,10} + e^{3,5,11,13} + e^{3,5,12,14} \\
& \quad + e^{4,6,7,9} + e^{4,6,8,10} + e^{4,6,11,13} + e^{4,6,12,14} + e^{7,8,9,10} \\
& \quad + e^{7,9,11,13} + e^{7,9,12,14} + e^{8,10,11,13} + e^{8,10,12,14} + e^{11,12,13,14} \big) \\
+ \, & e^{1,2} \big(e^{3,4,5} + e^{3,4,6} + e^{3,5,6} + e^{4,5,6} \\
& \quad + e^{7,8,9} + e^{7,8,10} + e^{7,9,10} + e^{8,9,10} \\
& \quad + e^{11,12,13} + e^{11,12,14} + e^{11,13,14} + e^{12,13,14} \big).
\end{aligned}$$

To see that the associated spectral sequence has nonzero d_5 differential, it suffices to show that $HC^\infty(\omega_Y + \omega_Y^{\circ 2}; \mathbb{F}_2) < \dim HC^\infty(\omega_Y; \mathbb{F}_2)$, for which it suffices to compute the rank of the two matrices. The rank of these matrices can be computed in Sage in a few seconds each to be 5600 and 5632, respectively. $\square$

Remark 3. The main theorem of [Lid13] is that $HF^\infty(Y)$ depends only on the triple cup product. Slightly more is true. The formulas for $\omega_Y^{\circ n}$, taken modulo k, depend only on the value of ω_Y modulo k. Combining this with the fact that $GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/k)$ is surjective, it follows that $HF^\infty(Y; \mathbb{Z}/k)$ depends only on the isomorphism class of the triple-cup product on $H^1(Y; \mathbb{Z})/k$. Note that this is less information than the triple-cup product on $H^1(Y; \mathbb{Z}/k)$.

This enables exhaustive searches. There are 348 3-forms on $\mathbb{F}_2^9$, considered up to isomorphism [HP21]. There are a bit over 3.5 million 3-forms on $\mathbb{F}_2^{10}$, considered up to isomorphism [KGP26]. Enumerating over all of these and comparing the cup homology groups shows that Ozsvath and Szabo's conjecture holds for $b_1(Y) \leqslant 10$, when working with $\mathbb{F}_2$ coefficients. Because $d_5 = 0$ over fields other than $\mathbb{F}_2$, one can argue that in fact the conjecture holds when $b_1(Y) \leqslant 10$ over all fields. It is plausible that the minimal counterexample has $b_1(Y) = 14$.

Statement on AI use. GPT 5.6 Sol produced a counterexample to Conjecture 1 with $R = \mathbb{F}_3$ and $b_1(Y) = 22$ of the form $e^1 \wedge \omega_1 + e^2 \wedge \omega_2$, where ω_i are two different symplectic forms on $\mathbb{F}_3^{20}$. This inspired a renewed search for simpler counterexamples over $R = \mathbb{F}_2$, which led to the counterexample above.

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